A refinement of the concept of equilibrium in multiple objective continuous games

Justo Puerto
Universidad de Sevilla

Rev.R.Acad.Cienc.Exact.Fis.Nat. (Esp)
Vol. 93, N.º 4, pp 457-462, 1999
Monográfico: Problemas complejos de decisión. II

A REFINEMENT OF THE CONCEPT OF EQUILIBRIUM IN MULTIPLE OBJECTIVE CONTINUOUS GAMES*

(Game theory/multiple objective games/vector programming/perfect equilibria)

J. Puerto, R. Infante and F. R. Fernández

Facultad de Matemáticas. Departamento de Estadística e Investigación Operativa. Universidad de Sevilla. 41012 Sevilla. Spain.

ABSTRACT

This paper considers a perfection refinement of the concept of equilibrium for multiple objective non-zero sum games. Based on the ideas of van Damme (1991) on perturbed games and stability the concept of perfect equilibrium is extended to a class of continuous games with multiple objetives. Existence is shown and several relationships that exist with the corresponding concept of scalar games are stated.

1. INTRODUCTION

Although nearly 40 years have passed the publication of Blackwell's paper (see Blackwell (1956)) which is the first known reference on Multiple Objetive Games (MOG) only a few papers have been devoed to this particular field among the wide literature of Game Theory. However, in recent years there has been some increasing interest in studying games with vector payoff. One of the reasons is that this approach represents better real-world situations of game theory. In fact, each competitive situation that can be moedeled as a scalar game has its counterpart as a multiple objective game when more than one scenario has to be compared simultaneously (see Fernández and Puerto (1996)).

Hash's (1951) concept of equilibrium is probably the most important solution concept in non-cooperative game theory. The notion of equilibrium in MOG was introduced and its existence proved by Shapley (1959) under restrictive hypotheses. The foundation behind this concept is that if one player does not specify an equilibrium as his strategy, then some player could gain by changing his strategy to something other than what was specified for him. Hence, no reason exists for players to play strategies that are not Nash's equilibrium. However, it is also well-known that any particular equilibrium does not have to be a reasonable prediction of reasonable behavior. We only can argue that any outcome that is not an equilibrium would necessarily be unreasonable as a description of how a player should behave. This fact leads several authors to considero refinements of Nash's equilibrium concept.

In recent years, some research has been devoted to study solution structures and algorithms for multicriteria games (see e.g. Bergstresser and Yu (1977) or Borm et al. (1988)). However, little attention has been focussed on the fundamental problem of existence of solutions. Wang (1993) dealt with this problem. In that paper, fixed-point theorems and other techniques are used to derive conditions for the existence of equilibria in games with vector payoffs. Novertheless, these equilibria are still not stable against small perturbations of all players' strategies.

Here we are interested in a class of n-person noncooperative MOG with uncountable set of strategies. We propose a refinement of the concept of equilibrium for those MOG based on the notion of perfectness. This refinement has been already proposed by Mendez-Naya et al. (1995) for Nash's equilibria in continuous scalar games. Moreover, Van Megen et al (1999) and Puerto and Fernández (1995, 1999) consider similar refinements for Nash equilibria of finite multiobjective games. By developing a solution theory that is based on such refinement we require the verification of some properties that would be theoretically desirable and, what is equally important, that the refined concept select a nonempty set of equilibria for any continuous MOG.

Acknowledgements: The research of the authors is partially supported by Spanish DG-ICYT gran number PB97-0707.

The paper is organized as follows. Section 2 is devoted to state the general setting where we formulate the MOG and introduces the concept of equilibrium for those games. Section 3 proposes the refinement of the concept of equilibrium. Section 4 presents some conclusions and the paper ends with an appendix where technical tools used in the paper are described.

2. THE CONCEPT OF EQUILIBRIUM IN MULTIPLE OBJETIVE GAMES

A continuous multiple objective game (MOG) in normal form is defined as a triplet $\Gamma = \{N, Y^i, u^i\}$ where $N = \{1, ..., n\}$ is the set of players. For each $i \in N$, $Y^i = [0, 1]$ is the set of pure strategies for player i. The player i's vector payoff u^i is a continuous function defined from u^i: $Y = \Pi_{i=1}^n Y^i \to \mathbb{R}^{m(i)}$ where $m(i)$ is the number of objectives of this player.

It is clear the MOG differs from single criterion (scalar) games only in the payoff functions. In MOG, each player has a vector payoffs to optimize, while in classical games they have scalar payoffs. In particular, if $m(1) = \cdots = m(n) = 1$ our game Γ becomes a n-person game in normal form.

oo Now, let us introduce the solution concept for a MOG problem. Let $\mathcal{P}(N)$ be the family of all non empty subset of N, i.e. the different coalitions of players in N.

For any $y = \{y^1, ..., y^n\} \in Y$ and $u = \{u^1, ..., u^n\} \in \mathbb{R}^{m(i)}$ where $y^i \in Y^i$ and $u^i = (u_1^i, ..., u_{m(i)}^i) \in \mathbb{R}^{m(i)}$, let $y_C = \{y^i : i \in C\}$ be the strategies of coalition C and $y_{-C} = y_{N/C}$ the strategies of players not in C. In the same way, let $u_C = \{u^i : i \in C\}$ and $u_{-C} = \{u^i : \notin C\}$ be the projections of u into $\mathbb{R}^C$ and $\mathbb{R}^{-C}$ respectively.

For each player i its set of mixed strategies S^i is the set of all the Borel-probability measures on [0, 1]. This is a subset of M the locally convex linear space of all the signed measured on [0, 1]. M is the dual of $C[0, 1]$ the space of all the continuous functions from [0, 1] into $\mathbb{R}$. S^i is a weakly* compact subset of M, hence compact in its weak* topology. As it is usual one can identify each measure $\mu \in M$ with the continuous linear functional $\langle \mu, f \rangle = \int f d\mu \ \forall f \in C[0, 1]$ (see the Appendix for more details on the weak* topology.)

A mixed strategy for the MOG Γ is a combination $s = (s^1, ..., s^n) \in S = (S^1 \times \cdots \times S^n)$. In the usual way, we can consider Y^i imbedded in S^i for all i because each $y^i \in Y^i$ corresponds to the mixed strategy which assigns probability 1 to y^i and 0 anywhere else. In the same way, any game with a finite number of strategies is also included in this framework. Once we have defined the mixed strategy space, we can extend the payoff functions.

$$u^i(s^1, ..., s^n) = \left(u_1^i(s^1, ..., s^n), ..., u_{m(i)}^i(s^1, ..., s^n)\right) \quad (1)$$

where $u_j^i(s^1, ..., s^n) = \int u_j^i ds^1 \dots ds^n$ for $j = 1, ... m(i)$ is the integral with respect to the product measure generated by $s^1, ..., s^n$. It is worth noting that u^i is a weakly* continuous function because the payoff functions u^i are continuous.

Let us introduce the concept of equilibrium point of the MOG Γ. We will use the notation $x = (y_C, y_{-C})$ for every coalition $C \in \mathcal{P}(N)$.

Definition 1. *A strategy $\bar{s} = (\bar{s}^1, ..., \bar{s}^n) \in S$ is an equilibrium point of $\Gamma = (N, Y^i, u^i)$, if for each $i \in N$, $\bar{s}^i$ is a weakly efficient solution of the vector maximum problem*

$$VM_i(s_{-i}) : \max_{s^i \in S^i} \left(u_1^i(s^i, \bar{s}_{-i}), ..., u_{m(i)}^i(s^i, \bar{s}_{-i})\right) \quad (2)$$

Hence, it is easy to understand that $\bar{s}$ is an equilibrium point if each player i chooses $\bar{s}^i$ as the best response to the strategy $\bar{s}_i$.

The above definition coincides with the conventional definition of continuous Nash equilibrium when $m(i) = 1$ for all i, i.e. in the particular case of scalar games. The interested reader can see the papers of Burger (1959) and Parthasaraty and Raghavan (1971) for further details and existence results in the scalar case and the paper of Wang (1993) for the multiobjective case.

For the purpose of the introduction of the idea of equilibrium in MOG any concept of efficiency would be valid. However, weak-efficiency induces the strongest dominance relation and therefore a player would only agree on deviation if the payoff in all the criteria increases (see e.g. Van Megen et al (1999)). In addition, to extend the concepts of stable equilibria of MOG we will need some topological properties as closedness of the whole set of equilibria that requires the use of weak efficiency rather than other efficiency concept.

Our first result states the existence of equilibrium points under general hypotheses. Let $\Lambda_{m(i)} = \{\lambda \in \mathbb{R}^{m(i)} : \lambda_j \geqslant 0, \Sigma_{j=1}^{m(i)} \lambda_j = 1\}$ $i = 1, ..., n$ be a set of weights. Let us consider then n-person unicriterion gam $\Gamma(\lambda)$, where $\lambda = (\lambda^1, ..., \lambda^n)$, $\lambda^i \in \Lambda_{m(i)}$ with the payoff functions $\hat{u}^i(s^i) = \Sigma_{j=1}^{m(i)} \lambda_j^i u_j^i(s^i,. s_{-i})$ for a s_{-i} fixed and $i = 1, ..., n$.

We state in the following lemma an straightforward result which is a consequence of the general theory of vector optimization.

Lemma 1. *Let Γ be a n-person MOG with continuous payoff functions u^i for all $i = 1, ..., n$ and strategy sets S^i $i = 1, ..., n$. Then $\bar{s} = (\bar{s}^1, ..., \bar{s}^n)$ is an equilibrium strategy of Γ iff there exists $\lambda = (\lambda^1, ..., \lambda^n) \in \Lambda_{m(1)} \times \cdots \times \Lambda_{m(n)}$ so that for all $i = 1, ..., n$ $\bar{s}^i$ is an optimal solution of the problem*

$$\max_{s^i \in S^i} \sum_{j=1}^{m(i)} \lambda_j^i u_j^i(s^i, \bar{s}_{-i}) \qquad (3)$$

Proof. Since each u^i is a continuous multi-linear function it is also concave in s^i whenever $\bar{s}_{-i}$ is fixed. Hence the result follows. □

The following theorem whose proof is direct from Lemma 1 characterizes the set of equilibrium points in MOG.

Theorem 1. *The set of equilibrium points of a MOG Γ in the conditions above coincides with the set of all Nash equilibria of games $\Gamma(\lambda)$ when λ varies in $\Lambda_{m(1)} \times \cdots \times \Lambda_{m(n)}$.*

Notice that this result is well-known and it already appears suggested in the paper of Shapley (1959)) although for finite two-person games.

3. A REFINEMENT OF THE CONCEPT OF EQUILIBRIUM

In order to introduce the refinement of the concept of equilibria of continuous MOG we use the technique of perturbing the set of admissible strategies and define the refined equilibria as limits of sequences of equilibria in the modified games. It should be noted that these ideas were firstly applied by Selten (1975) when he defined the concept of perfect equilibria in scalar games. The goal of this section is to extend the above mentioned concept of perfectness to continuous MOG.

In an equilibrium, each player's equilibrium strategy is an efficient response to the other players' equilibrium strategies. In a perfect equilibrium, there must also be arbitrarily small perturbations of all players' strategies such tha every pure strategy gets strictly positive probability and each player's equilibrium is still an efficient response to the other players' perturbed strategies. Thus, as in Selten's perfect equilibrium we discriminate solutions which are not stable against any arbitrarily slight perturbation of the game strategies.

A perturbation or error vector is a measure $\mu = (\mu^1, ..., \mu^n) \in \Pi_{i=1}^n S^n$, satisfying for all i:

1. $\mu^i((a, b]) > 0$ for any interval $(a, b] \subseteq [0, 1]$ with $a < b$.
2. $\mu^i([0, 1]) < 1$.

The trembling-hand strategies associated to a perturbation vector μ are the probability measures in $X^1(\mu^1) \times \cdots \times X^n(\mu^n)$ where

$$X^i(\mu^i) = \{F^i \in S^i : F^i(I) \geqslant \mu^i(I) \ \forall I = (a, b] \subseteq [0, 1]\} \quad (4)$$

A mixed strategy $\bar{s} \in S$ is an equilibrium in an μ-perturbed game if for all $i = 1, ..., n$, $\bar{s}^i$ is a weakly efficient solution of

$$P_i(\mu^i, \bar{s}) : \max \left(u_1^i(s^i, \bar{s}_{-i}), ..., u_{m(i)}^i(s^i, \bar{s}_{-i})\right) \quad (5)$$
$$s^i \in X^i(\mu^i)$$

With these preliminaires we are in position to introduce the concept of perfect equilibrium in MOG.

Definition 2. *We say that $s = (s^1, ..., s^n) \in S$ is a perfect strategy combination of the MOG Γ if there exists a sequence $\{\mu_k\}$ of perturbation vectors and a sequence $\{s_k\}$ of strategies with $s_k = (s_k^1, ..., s_k^n)$ such that:*

1. $\mu_k \xrightarrow{*} 0$;
2. *s_k is an equilibrium in an μ_k-perturbed game for all k;*
3. $s_k^i \xrightarrow{*} s^i, \forall i = 1, ..., n$.

It should be noticed that when $m(i) = 1$ for all $i = 1, ..., n$ our perfect equilibrium reduces to the concept of perfect equilibrium for continuous games as introduced in Mendez-Naya et al (1995).

First of all, we prove that every perfect equilibrium is also an equilibrium. Then, we show that the converse is not true. To prove the inclusion we need some technical results.

Let $Y^i(\mu^i, \bar{s}_{-i})$ be the set of all the values of the payoff function $u^i(s, \bar{s}_{-i})$ whenever $s^i \in X^i(\mu^i)$. This is

$$Y^i(\mu^i, \bar{s}_{-i}) = \{y \in \mathbb{R}^{m(i)} : y = u^i(s, \bar{s}_{-i}), s \in X^i(\mu^i)\}.$$

This family of sets induces a point-to-set map:

$$Y^i : S \hookrightarrow 2^{\mathbb{R}^{m(i)}}$$
$$\mu^i \hookrightarrow Y^i(\mu^i, \bar{s}_{-i}) \quad (6)$$

This application is continuous in the sense of Hogan (1973) (see the Appendix for details). Continuity of Y^i is used to prove the existence of perfect equilibria as it will be shown in the following. To prove this property we need to prove previously the continuity of the map X defined by (4).

Lemma 2. *The point-to-set map X defined by*

$$X : M \hookrightarrow 2^{M}$$
$$\mu \hookrightarrow X(\mu) \quad (7)$$

is continuous.

Proof. First, we prove that S is upper semicontinuous (u.s.c.). Indeed, let $\{\mu_k\} \subseteq M$ be a sequence which converges in the weak* topology to μ, this is $\mu_k \xrightarrow{*} \mu$, and let $F_k \in X(\mu_k)$ for all k satisfying $F_k \xrightarrow{*} F$. Consider the weak* continuous function g defined by:

$$\begin{aligned} g: M \times M &\longrightarrow M \\ (\mu, F) &\longrightarrow \mu - F \end{aligned} \tag{8}$$

Then, $g(\mu_k, F_k) \xrightarrow{*} g(\mu, F) = \mu - F$. Moreover, since $F_k \in X(\mu_k)$ for all k then $(\mu_k - F_k)(I) \leqslant 0$ for all $I = (a, b] \subseteq [0, 1]$. Hence $(\mu - F)(I) \leqslant 0$ for all $I = (a, b] \subseteq [0, 1]$ and $F \in X(\mu)$ which proves that X is u.s.c.

Now, we prove that X is lower semicontinuous (l.s.c.). We have to prove that given $\{\mu_k\} \subseteq M$ a sequence which satisfies $\mu_k \xrightarrow{*} \mu$, and $F \in X(\mu)$ then there exists $F_k \in X(\mu_k)$ except for a finite number of k, satisfying $F_k \xrightarrow{*} F$. We distinguish two cases.

1) $F(I) = \mu(I)$ for any $I = (a, b] \subseteq [0, 1]$. In this case, we can take $F_k = \mu_k$ and the result follows.

2) $F \neq \mu$. Since μ is a finite measure, there exists a measure $\hat{F} \in M$ such that $F(I) > \hat{F}(I) > \mu(I)$ if $F(I) > \mu(I)$ and $F(I) > \mu(I)$ if $F(I) = \mu(I)$. Consider the sequence $F_k = (1 - \lambda_k)F + \lambda_k\hat{F}$ with $\lambda_k \in [0, 1]$. It is clear that $F_k \xrightarrow{*} F$ if $\lambda_k \longrightarrow 0$. On the other hand, $F_k \in X(\mu_k)$ if $F_k - \mu_k \geqslant 0$ which is equivalent to $\lambda_k(\hat{F} - F) + F - \mu_k \geqslant 0$. Now, for any $I = (a, b] \subseteq [0, 1]$ if $F(I) > \mu(I)$ then $(F_k - \mu_k)(I) \geqslant 0$ if λ_k is small enough. Otherwise, this is if $F(I) = \mu(I)$ then $(F_k - \mu_k)(I) \geqslant 0$ for any λ_k. Hence, if $\lambda_k \to 0$ then:

$$F_k \xrightarrow{*} F$$
$$F_k \in X(\mu_k)$$

and the proof is complete. □

Lemma 3. *The point-to-set map Y^i defined by (6) is continuous at $\hat{\mu}$ for any $i = 1, ..., n$.*

Proof. Let $\{\mu_k\} \subset M$, $\mu_k \xrightarrow{*} \hat{\mu}$ and $y_k \in Y^i(\mu_k, \bar{s}_{-i})$ for all k, $y_k \to \hat{y}$. From the definition of $Y^i(\mu_k, \bar{s}_{-i}$ there exists $\{F_k\} \subset X^i$ such that $F_k \in X^i(\mu_k)$ and $y_k = u(F_k, \bar{s}_{-i})$. Since S^i is weak* compact the sequence $\{F_k\}$ (or some subsequence) converges in the weak* topology to some $\hat{F}$. Then, $\hat{F}$ belongs to $X^i(\hat{\mu})$ because X^i is u.s.c... Now, since u is weak* continuous $u(F_k, \bar{s}_{-i}) \to u(\hat{F}, \bar{s}_{-i})$. This implies that $\hat{y} = u(\hat{F}, \bar{s}_{-i}) \in Y^i(\hat{\mu}, \bar{s}_{-i})$. Hence Y^i is u.s.c.

Let $\{\mu_k\} \subset M$, $\mu_k \xrightarrow{*} \hat{\mu}$ and $\hat{y} \in Y^i(\hat{\mu}, \bar{s}_{-i})$. By definition of Y^i there exists $\hat{F} \in X^i(\hat{\mu})$ such that $\hat{y} = u(\hat{\mu}, \bar{s}_{-i})$. Now, since X^i is l.s.c. there exists a sequence $\{F_k\}$ with $F_k \in X^i(\mu_k)$ except for a finite number of k such that $F_k \xrightarrow{*} \hat{F}$. Take $y_k = u(\hat{\mu}_k, \bar{s})$, then $y_k \in Y^i(\mu_k, \bar{s}_i)$. Moreover, the continuity of u implies that $y_k \longrightarrow \hat{y}$. This proves that Y^i is l.s.c. at $\hat{\mu}$. □

Let $N^i(\mu, \bar{s}_{-i})$ be the set of weakly efficient solutions of $P_i(\mu, \bar{s})$ on the image space. This is

$$N^i(\mu, \bar{s}_{-i}) = \{y \in Y^i(\mu, \bar{s}_{-i}) : \nexists y' \in Y^i(\mu,. \bar{s}_{-i}) \text{ with } y' > y \text{ componentwise}\}.$$

Now, we consider the point-to-set map defined by the bove introduced family of sets.

$$\begin{aligned} N^i : M &\hookrightarrow 2^{\mathbb{R}^{m(i)}} \\ \mu &\hookrightarrow N^i(\mu^i, \bar{s}_{-i}) \end{aligned} \tag{9}$$

Lemma 4. *The point-to-set map N^i is u.s.c. at $\hat{\mu}$.*

Proof. Apply Theorem 4.2.1 in Sawaragy et al. (1985) taking into account that in our case the domination structure $D = \mathbb{R}^{m(i)}_{-}$ is constant and convex and that by Lemma 3 Y^i is a continuous point-to-set map. □

Let us consider the point-to-set map M^i defined as

$$M^i(\mu, \bar{s}_{-i}) = \{s^i \in X^i(\mu) : u^i(s^i, \bar{s}_{-i}) \in N^i(\mu, \bar{s}_{-i})\} \tag{10}$$

We state the following lemma which is used in the next theorem. This is a consequence of the previous results. However, for the sake of completeness a proof is given.

Lemma 5. *The point-to-set map M^i defined in (10) is an upper semicontinuous map at $\hat{\mu}$, for any $i = 1, ..., n$.*

Proof. Let

$$\{\mu_k\} \subset S,\ \mu_k \xrightarrow{*} \hat{\mu},\quad F_k \in M^i(\mu_k, \bar{s}_{-i}), \text{ and } F_k \xrightarrow{*} \hat{F}.$$

Since X^i is u.s.c. at $\hat{\mu}$, $\hat{F} \in X^i(\hat{\mu})$. From the definition of $M(\mu, \bar{s}_{-i})$, $u(F_k, \bar{s}_{-i}) \in N^i(\mu_k, \bar{s}_{-i})$. From the weak* continuity of u:

$$u(F_k, \bar{s}_{-i}) \longrightarrow u(\hat{F}, \bar{s}_{-i}).$$

Hence, $u(\hat{F}, \bar{s}_{-i}) \in N^i(\hat{\mu}, \bar{s}_{-i})$, since the map N^i is u.s.c. at $\hat{\mu}$. Therefore, $\hat{F} \in M^i(\hat{\mu}, \bar{s}_{-i})$, and so the map M is u.s.c. at $\hat{\mu}$. □

Let us assume the hypothesis of Lemma 5 then we have the following result which states the relation between the set of equilibria and perfect equilibria of a MOG.

Theorem 2. *For any continuous MOG in normal form, the set of perfect strategy combinations is a subset of the set of equilibria.*

Proof. Let us consider a perfect equilibrium s and let $\{s_k\}$ and $\{\mu_k\}$ be two sequences such that s_k is an equilibrium in an μ_k-perturbed game for all k and verifying that $(s_k, \mu_k) \xrightarrow{*} (s, 0)$. Hence, we have for all $i = 1, ..., n$ that $s_k^i \in M^i(\mu_k^i\ (s_{-i})_k)$ then the u.s.c. of M^i implies that $s^i \in M^i(0, s_{-i})$ for all $i = 1, ..., n$. That is, s^i is a weakly efficient solution of the problem $VM_i(s_{-i})$ for all $i = 1, ..., n$ what by Definition 1 implies that s is an equilibrium point. □

In the following, we provide an example which shows that the concept of perfect equilibrium is a strict refinement of the concept of equilibrium in continuous multiobjective games.

Example 1. Let us consider tha two-persons two objectives continuous game Γ with vector payoffs given by:

$$u^1(x, y) = u^2(x, y) = (xy, x^2y) \quad \text{for all } (x, y) \in [0, 1] \times [0, 1].$$

The game Γ induces two continuous single objective games Γ_i with $i = 1, 2$ with the payoff functions u_i^1, u_i^2 defined on [0, 1] × [0, 1] by:

$$u_1^1(x, y) = u_1^2(x, y) = xy$$
$$u_2^1(x, y) = u_2^2(x, y) = x^2y.$$

It is clear that any equilibrium (F, G) in Γ is also an equilibrium in Γ_i $i = 1, 2$. Indeed, if we assume that (F, G) is not an equilibrium in Γ_1 then it must exist F^* such that:

$$\int xy\ dF^*(x)dG(y) < \int xy\ dF(x)dG(y). \qquad (11)$$

Now, since for any $(x_1, y_1), (x_2, y_2) \in [0, 1] \times [0, 1]$ we have that $x_1y_1 \leqslant x_2y_2$ if and only if $x_1^2y_1 \leqslant x_2^2y_2$ then (11) implies that:

$$\int x^2y\ dF^*(x)dG(y) < \int x^2y\ dF(x)dG(y). \qquad (12)$$

Thus,

$$u^1(F^*, G) = \left(\int xy\ dF^*(x)dG(y), \int x^2y\ dF^*(x)dG(y)\right)$$
$$< \left(\int xy\ dF(x)dG(y), \int x^2y\ dF(x)dG(y)\right) =$$
$$= u^1(F, G).$$

This last inequality means that (F, G) would not be an equilibrium in Γ because F^* is a better response to G than F. This contradiction proves that any equilibrium in Γ is also an equilibrium in the associated single objective games Γ_1, Γ_2. Notice that the same argument can be applied to the μ_k-perturbed games.

Let us denote by $\delta(x)$ the degenerate probability measure which assigns probability 1 to x and 0 everywhere else. It is clear that $(\delta(0), \delta(0))$ is a multiobjective equilibrium of the game Γ. Assume that it is perfect. Then, there exists a sequence $\{(F^k, G^k)\}_{k\geqslant 1}$ of Nash equilibria of the corresponding sequence of μ^k-perturbed games, converging in the weak* topology to $(\delta(0), \delta(0))$. It is clear that $u^1(\delta(1), G^k) > u(F, G^k)$ (componentwise) for all $k \geqslant 1$ (provided that $F \neq \delta(1)$). Then, by application of Theorem 3 in Mendez-Naya et al. (1995) to the single objective games Γ_i $i = 1, 2$, it follows that F^k coincides with μ^k on [0, 1). Therefore, since F^k is a probability measure, when μ^k goes to zero F^k converges to $\delta(1)$. This contradicts that $(\delta(0), \delta(0))$ is the limit of the sequence $\{(F^k, G^k)\}_{k\geqslant 1}$. Hence $(\delta(0), \delta(0))$ is an equilibrium which is no perfect. △

Finally, the existence of such perfect equilibrium points is stated in the following theorem.

Theorem 3. *For any continuous MOG in normal form Γ, there exists at least one perfect equilibrium.*

Proof. Since each $X^i(\mu_k)$ is a weak* compact, convex set and the objective functions $u^i(\cdot, \bar{s}_{-i})$ are linear for any $\bar{s}_{-i}$ fixed then there exists s_k being an equilibrium of the μ_k-perturbed game (see e.g. Corollary 3.2 in Wang (1993)).

Now, $\{s_k\}_{k\geqslant 1}$ is a sequence in $S^1 \times \cdots \times S^k$ which is a weak* compact set. Then, there exists a weak* convergent subsequence $\{s_{n_k}\}_{k\geqslant 1}$ included in $\{s_k\}_{k\geqslant 1}$. Hence, applying the upper semicontinuity of M^i for all $i = 1, ..., n$ the result follows. □

4. CONCLUDING REMARKS

In this paper the concept of perfect equilibrium in continuous MOG is introduced as a refinement of the concept of equilibrium in MOG. To this end, we have used the approaches followed by van Damme (1991) and Mendez-Naya et al. (1995) based on prtubed games.

Our first remark is on the developement of procedures that compute these kinds of equilibria. At first sight, it seems to be a hard task because it is even harder than in the scalar case. Thus, more research should be directed towards this particular point.

Secondly, we would point out that the basic rationale for considering perturbed model with small probability of errors is that they give us a way to test the principle that an equilibrium does not depend on the unreasonable assumption that players ignore the pure strategies of the game having zero probability in the equilibrium. Hence, when we use this approach we are only using part of the basic rationality principles about players' rational behavior in a multiple objective game. Unfortunately, we think that as it happens in the scalar case there is no solution concept that verifies all the rationality principles of intelligent behavior in MOG.

APPENDIX

An interestig and important notion that arises naturally when we consider the mixed strategies in the continuous games is the weak* topology. Starting with $C[0, 1]$ the normed space of the continuous functions in $[0, 1]$ we form its topological dual M the set of all the Borel signed measures on $[0, 1]$.

Definition 3. *A sequence $\{s_n\} \subset M$ is said to be weak* convergent to an element $s \in M$ if for every $f \in C[0, 1]$ $\langle f, s_n \rangle \to \langle x, s \rangle$. In this case we write $s_n \xrightarrow{*} s$.*

Based on this convergence it is possible to consider a notion of compactness (less severe than the usual one) but still sufficient to provide alternative explanation for the existence of solutions to optimization problems.

Definition 4. *A set $K \subseteq M$ is said to be weak* compact if every infinite sequence from K contains a weak* convergent subsequence.*

Separable normed linear spaces posses an important propertyu characterizing this kind of compactness.

Lemma 6. *Let X be a seprable, normed linear vector space and X^* its topological dual. Every bounded sequence in X^* contains a weak* convergent subsequence.*

A well-known consequence of this result is that the closed bounded sets in X^* are weak* compact. Since $C[0, 1]$ is a separable space these results apply.

Definition 5. *A functional L defined on M is said to be weak* continuous at $s_0 \in M$ if given $\varepsilon > 0$ and a finite collection $\{f_1, f_2, ..., f_n\}$ from $C[0, 1]$ sduch that $|L(s) - L(s_0)| < \varepsilon$ for all $s \in M$ such that $|\langle s - s_0, f_i \rangle| < \delta$ for all $i = 1, ..., n$.*

In the following, we recall some results concerning the stability of optimal solution sets of multiobjective optimization problems regarding perturbations of feasible solution sets. A point-to-set map or multimapping F from a set X into a set Y is a map that associates a subset of Y with each point of X.

In what follows, we introduce several concepts with regards to the continuity of these maps. We follow the definitions of Hogan (1973). For additional details on this and other subjects we refer the interested readers to the book of Sawaragi et al. (1985).

Let F be a point-to-set map from a set X into a set Y.

Definition 6. *F is said to be.*

1. *lower semicontinuous (l.s.c.) a point $x \in X$ if $\{x^k\} \subset X$, $x^k \to x$, and $y \in F(x)$ all imply the existence of an integer m and a sequence $\{y^k\} \subset Y$ such that $y^k \in F(x^k)$ for $k \geqslant m$ and $y^k \to y$;*

2. *upper semicontinuous (u.s.c.) at point $x \in X$ if $\{x^k\} \subset X$, $x^k \to x$, $y^k \in F(x^k)$ and $y^k \to y$ all imply that $y \in F(x)$;*

3. *continuous at point $x \in X$ if it is both l.s.c. and u.s.c. at x.*

REFERENCES

1. Blackwell, O. (1956). An analog of minimax theorem for vector payoffs. *Pacific Journal of Mathematics,* **6,** 1:1-8.
2. Bergstresser, K. and Yu, P. L. (1977). Domination Structures and Multicriteria Problems in *N*-person Games. *Journal of Theory and Decision,* **8,** 5-48.
3. Borm, P., Tijs and Van den Arsen, J. (1988). Pareto equilibria in multiobjective games. *Methods of Operations Research* **60,** 302-312.
4. Fernández, F. R. and Puerto, J. (1996). Vector Linear Programming in Zero-Sum Multicriteria Matrix Games. *Journal of Optimization Theory and Applications* **89,** 115-127.
5. Hogan, W. (1973). Point-to-set maps in mathematical programming. *SIAM Rev.,* **15,** 591-603.
6. Van Megen, F. Borm, P. and Tijs, S. (1999). A perfectness concept for multicriteria games. *Mathematical Methods of OR* **49,** 3:401-412.
7. Méndez-Naya, L., García-Jurado, I. and Ceso, J. C. (1995). Perfection of Nash equilibria in continuous games. *Mathematical Social Sciences* **29,** 225-237.
8. Nash, J. F. (1951). Non-cooperative games. *Annals of Mathematics* **54,** 286-295.
9. Parthasarathy, T. and Raghavan, T. E. S. (1971). *Some topics in two-persons games.* Elsevier, New York.
10. Puerto, J. and Fernández, F. R. (1995). *Solution Concepts based on security levels in Constrained Multicriteria Convex Games. Opsearch* **32,** 16-30.
11. Puerto, J. and Fernández, F. R. (1999). A Refinement of the Concept of Equilibrium in Multiple Obective Games. Nonlinear Analysis and Convex Analysis, Wataru Takahashi and Tamaki Tanaka (eds.), 313-320, World Scientific publishers.
12. Sawaragi, Y., Nakayama, H. and Tanino, T. (1985). *Theory of Multiobjective Optimization.* Academic Press. Orlando, Florida.
13. Selten, R. (1975). Reexamination of the perfectness concept for equilibrium points in extensive games. *Int J Game Theory* **4,** 25-55.
14. Shapley, L. S. (1959). Equilibrium Points in Games with Vector Payoffs. *Naval Research Logistics Quarterly* **6,** 1:57-61.
15. Van Damme, E. (1991). *Stability and perfection of Nash equilibria.* Springer Verlag, Berlin.
16. Wang, S. (1993). Existence of a Pareto equilibrium. *Journal of Optimization Theory and Application* **3**, 61-63.